Kuwait University

Department of Mathematics and Computer Science

Math. 102, Calculus II — Date: 24/03/2007

First Exam. — Duration: 90 min.

Answer all questions. Calculators and Mobile Phones are not allowed.

1. ((1+ 2+3) pts.) Let $f(x) = \tan^{-1}(\ln x) + \frac{\pi}{2}$.

(a) What is the domain of $f(x)$?

(b) Show that f has an inverse function.

(c) Find $f^{-1}(x)$ and its domain.

2. (2 pts.) Calculate $\sin(\frac{\pi}{2} + \sec^{-1}(2))$.

3. ((3 + 2) pts.)

(a) Find $\frac{dy}{dx}$ if $y = (\sin x)^{\cosh x} + \pi^{\sec^{-1} x} + e^{\pi}$.

(b) Use implicit differentiation to find $\frac{dy}{dx}$ if $\tan^{-1}(y) = \ln|1 + xy|$.

4. (3 pts.) Prove the following

$$\operatorname{sech}(\ln x) + \operatorname{csch}(\ln x) = \frac{4x^3}{x^4 - 1}$$

5. (3 pts each.) Evaluate the following integrals

$$(a) \quad \int \frac{1}{x\sqrt{9x^4 - 16}}\, dx.$$

$$(b) \quad \int \frac{1}{e^{2x} + e^{-2x}}\, dx.$$

$$(c) \quad \int \frac{\sec(x)}{\cos(x) + \sin(x)}\, dx.$$

1) $f(x) = \tan^{-1}(\ln x) + \pi/2 \quad D_f: \{x : -\infty < \ln x < \infty \Rightarrow 0 < x < \infty\}$

$y = \lim_{x \to 0} \tan^{-1}(\ln x) + \pi/2 = -\pi/2 + \pi/2 = 0, \quad \lim_{x\to\infty} \tan^{-1}(\ln x) + \frac{\pi}{2} = \pi$

$R_f \ (0, \pi)$.

b) $f'(x) = \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x} > 0$ in domain $\uparrow$ $\therefore$ 1-1 fn Inverse exists

c) $y = \tan^{-1}(\ln x) + \pi/2 \Rightarrow y - \frac{\pi}{2} = \tan^{-1}(\ln x) = \tan(y - \frac{\pi}{2}) = \ln x \Rightarrow -\cot y$.

$\therefore x = e^{-\cot y} \quad \therefore f^{-1}(x) = e^{-\cot x}$ Ans

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Q2 $\sin(\frac{\pi}{2} + \sec^{-1} 2) = \sin(\frac{\pi}{2} + \frac{\pi}{3}) = \sin(\frac{3\pi + 2\pi}{6}) = \sin\frac{5\pi}{6} = \frac{1}{2}$ Ans.

Q3 a) $y = (\sin x)^{\cos x} + \pi^{\sec^{-1} x} + e^{\pi} \Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x}\left[\cos x \frac{1}{\sin x} \cos x + \ln \sin x (-\sin x)\right] + \pi^{\sec^{-1} x} \ln \pi \cdot \frac{1}{x\sqrt{x^2-1}}$

$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\cot x \cos x + \sin x \ln(\sin x)\right] + \frac{1}{x\sqrt{x^2-1}} \pi^{\sec^{-1} x} \ln \pi$ Ans

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b) $\tan^{-1} y = \ln(1+xy) \Rightarrow \frac{1}{1+y^2} y' = \frac{1}{(1+xy)}[xy' + y] = \frac{y'}{1+y^2} - \frac{x}{1+xy} y' = \frac{y}{1+xy}$

$\Rightarrow y' = \dfrac{\frac{y}{1+xy}}{\frac{1}{1+y^2} - \frac{x}{1+xy}} = \dfrac{y(1+y^2)}{1+xy - x - xy^2}$ Ans

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Q4 $\sec(\ln x) + \csc(\ln x) = \frac{4x^3}{x^4-1}$

L.H.S $\frac{2}{e^{\ln x} + e^{-\ln x}} + \frac{2}{e^{\ln x} - e^{-\ln x}} = \frac{2}{x + \frac{1}{x}} + \frac{2}{x - \frac{1}{x}} = 2\left[\frac{x}{x^2+1} + \frac{x}{x^2-1}\right]$

$= 2x\left[\frac{x^2 - 1 + x^2 + 1}{x^4-1}\right] = \frac{4x^3}{x^4-1}$ R.H.S

Q5 a) $\frac{1}{6}\int \frac{6 \cdot x\, dx}{x^2\sqrt{(3x^2)^2 - 4^2}}$ Let $3x^2 = u \Rightarrow x^2 = \frac{u}{3} \therefore 6x\,dx = du$

$\therefore \frac{1}{6}\int \frac{du}{\frac{u}{3}\sqrt{u^2-4^2}} = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-4^2}} du = \frac{1}{2} \cdot \frac{1}{4} \sec^{-1}\frac{u}{4} + C = \frac{1}{8}\sec^{-1}\frac{3x^2}{4} + C$

b) $\int \frac{e^{2x}}{(e^{2x})^2 + 1} dx = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} e^{2x} + C$

c) $\int \frac{\sec x}{\cos x + \sin x} dx = \int \frac{\sec^2 x}{1 + \tan x} dx = \ln|1 + \tan x| + C$

Taslim